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2001 J. Phys. A: Math. Gen. 34 7901

(http://iopscience.iop.org/0305-4470/34/38/314)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 7901–7916 PII: S0305-4470(01)18392-7

Phase observables, phase operators and operator orderings

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Received 27 October 2000, in final form 10 July 2001
Published 14 September 2001
Online at stacks.iop.org/JPhysA/34/7901

Abstract

We represent quantum phase observables as phase shift covariant normalized positive operator measures. The phase operators are the first moment operators of the phase observables. A phase operator determines the associated phase observable uniquely. We show that the Cahill–Glauber s-ordered phase operators are determined by phase shift covariant generalized operator measures, which are ordinary operator measures whenever $\mathrm{Re}\, s < 0$ and phase observables when $s \leqslant -1$. The Wigner–Weyl quantized phase operator is not determined by any phase observable. We investigate the classical limit of covariant (generalized) operator measures in coherent states.

PACS numbers: 03.65.−w, 02.03.−f, 05.70.Fh, 42.50.Dv

1. Introduction

In the textbook presentation of quantum mechanics, a physical quantity, an observable, is described by a self-adjoint operator or, equivalently, by a spectral measure on a Hilbert space. It is well known that this description is too narrow and that its natural extension is given by the concept of a positive operator measure. In this wider interpretation, quantum phase observables are represented as phase shift covariant normalized positive operator measures [1–3].

The traditional quantization rules associate self-adjoint operators to classical dynamical variables. In the case of phase, the natural dynamical variable to be quantized is the angle variable of the two dimensional phase space (for an overview, see e.g. [4] or [5]). In this paper, we investigate the possibility to represent some of such quantized phase angles as the first moment operators of phase observables.

The first moment operator defines the related phase observable uniquely. Thus, it is natural to consider it as a phase operator. Conversely, to call a bounded self-adjoint operator a phase operator, it should be the first moment operator of a phase observable, since the spectral measure of such an operator is never phase shift covariant and therefore cannot be considered as a phase observable. For example, the Wigner–Weyl quantized phase angle has been suggested to be a phase operator [6–9] but, as we will see, it is not determined by any phase observable.

Cahill and Glauber [10] defined the s-ordering rule which can be used to quantize the phase angle and to get the so-called s-ordered phase operators. We will show that this quantization leads to phase operators only for some values of the parameter s. Still, the non-positive covariant normalized generalized operator measures determined by the s-ordered phase operators behave well in the classical limit of coherent states.

2. Phase observables

2.1. The basic theorem

Let $\mathcal{H}$ denote a complex separable Hilbert space with a fixed basis $\{|n\rangle \in \mathcal{H} | n \in \mathbb{N}\}$ and let $\mathcal{L}(\mathcal{H})$ denote the set of bounded operators on $\mathcal{H}$. For any $A \in \mathcal{L}(\mathcal{H})$ we write $A_{n,m}$ instead of $\langle n|A|m\rangle$ to shorten the notations. Define the lowering and the number operators as $a := \sum_{n=0}^{\infty} \sqrt{n+1}|n\rangle\langle n+1|$ and $N := a^*a = \sum_{n=0}^{\infty} n|n\rangle\langle n|$, respectively, with their usual domains $\mathcal{D}(a) := \{\phi \in \mathcal{H} \mid \sum_{n=0}^{\infty} n|\langle n|\phi\rangle|^2 < \infty\}$ and $\mathcal{D}(N) := \{\phi \in \mathcal{H} \mid \sum_{n=0}^{\infty} n^2|\langle n|\phi\rangle|^2 < \infty\}$, and let $R(\theta) := \mathrm{e}^{\mathrm{i}\theta N}$, $\theta \in \mathbb{R}$, be a phase shifter.

Let Ω be a Borel subset of the complex plane $\mathbb{C}$ and let $\mathcal{B}(\Omega)$ denote the σ-algebra of the Borel subsets of Ω. We say that a mapping $E: \mathcal{B}(\Omega) \to \mathcal{L}(\mathcal{H})$ is an *operator measure* if it is σ-additive (in the weak operator topology) and that E is *normalized* if $E(\Omega) = I$. If $E(X) = E(X)^*$ or $E(X) \geqslant O$ for all $X \in \mathcal{B}(\Omega)$ we say that E is *self-adjoint* or *positive*, respectively. The fact that an operator measure $E: \mathcal{B}(\Omega) \to \mathcal{L}(\mathcal{H})$ is normalized and positive equals the fact that for any unit vector ψ the mapping $X \mapsto \langle\psi|E(X)\psi\rangle$ is a probability measure. A positive normalized operator measure (POM) $E: \mathcal{B}(\Omega) \to \mathcal{L}(\mathcal{H})$ is a *spectral measure* if $E(X)^2 = E(X)$ for all $X \in \mathcal{B}(\Omega)$.

An operator measure $E^\alpha: \mathcal{B}([\alpha-\pi, \alpha+\pi)) \to \mathcal{L}(\mathcal{H})$, where $\alpha \in \mathbb{R}$, is *covariant* under the phase shifts generated by the number operator N if

$$R(\theta)E^\alpha(X)R(\theta)^* = E^\alpha(X \oplus \theta) \tag{1}$$

for all $X \in \mathcal{B}([\alpha-\pi, \alpha+\pi))$ and for all $\theta \in [0, 2\pi)$, where $X \oplus \theta := \{x \in [\alpha-\pi, \alpha+\pi) \mid (x-\theta)(\mathrm{mod}\ 2\pi) \in X\}$. A *phase observable* is a covariant POM $E: \mathcal{B}([0, 2\pi)) \to \mathcal{L}(\mathcal{H})$. An important example of phase observables is the *canonical phase observable* [1, 2, 11]. It can be defined by using the London phase states

$$|\theta) := \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \mathrm{e}^{\mathrm{i}n\theta}|n\rangle$$

as

$$E_{\mathrm{can}}(X) := \int_X |\theta)(\theta|\,\mathrm{d}\theta = \sum_{n,m=0}^{\infty} \frac{1}{2\pi} \int_X \mathrm{e}^{\mathrm{i}(n-m)\theta}\,\mathrm{d}\theta|n\rangle\langle m| \qquad X \in \mathcal{B}([0, 2\pi)). \tag{2}$$

Next we generalize the definition of a phase observable.

We note that any phase interval $[\alpha-\pi, \alpha+\pi), \alpha \in \mathbb{R}$, is an equally good set of measurement outcomes of a phase observable as $[0, 2\pi)$ since we only change the parametrization of the unit circle. However, as we will see in section 3, the first moment operators of different covariant POMs based on intervals $[\alpha-\pi, \alpha+\pi)$ and $[\beta-\pi, \beta+\pi)$, $\alpha \neq \beta$, differ. Thus, for all $\alpha \in \mathbb{R}$ we define an α-*shifted phase observable* E^α as a covariant POM based on the α-shifted interval $[\alpha-\pi, \alpha+\pi)$.

Following the proof of phase theorem 2.2 of [3] one gets the structure theorem of covariant normalized (not necessarily positive) operator measures:

Theorem 2.1. *Fix* $\alpha \in \mathbb{R}$ *and let* $E^\alpha : \mathcal{B}([\alpha - \pi, \alpha + \pi)) \to \mathcal{L}(\mathcal{H})$ *be a covariant normalized operator measure. For any* $X \in \mathcal{B}([\alpha - \pi, \alpha + \pi))$,

$$E^\alpha(X) = \sum_{n,m=0}^{\infty} c_{n,m} \frac{1}{2\pi} \int_X \mathrm{e}^{\mathrm{i}(n-m)\theta}\, \mathrm{d}\theta |n\rangle\langle m| \tag{3}$$

where the series converge in the weak operator topology (and the summation order is irrelevant), and where $c_{n,m} \in \mathbb{C}$ *and* $c_{n,n} = 1$ *for all* $n, m \in \mathbb{N}$*. If* E^α *is self-adjoint then* $\overline{c}_{n,m} = c_{m,n}$.

We say that the matrix $(c_{n,m})_{n,m\in\mathbb{N}}$ of theorem 2.1 is the *structure matrix* of E^α. If E^α is an α-shifted phase observable we say that $(c_{n,m})_{n,m\in\mathbb{N}}$ is the *phase matrix* of E^α. For any α-shifted phase observable E^α the phase matrix $(c_{n,m})$ is positive semidefinite, that is, all the principal minors of $(c_{n,m})$ are non-negative (see [11]). In particular, this implies that $|c_{n,m}| \leqslant 1$ for all $n, m \in \mathbb{N}$. Conversely, if $(c_{n,m})$ is a positive semidefinite complex matrix with the diagonal elements equal to one then the map

$$E^\alpha : \mathcal{B}([\alpha - \pi, \alpha + \pi)) \to \mathcal{L}(\mathcal{H})$$
$$X \mapsto E^\alpha(X) := \sum_{n,m=0}^{\infty} c_{n,m} \frac{1}{2\pi} \int_X \mathrm{e}^{\mathrm{i}(n-m)\theta}\, \mathrm{d}\theta |n\rangle\langle m|$$

defines an α-shifted phase observable for all $\alpha \in \mathbb{R}$ (see phase theorem 2.2 of [3]). This shows that there is a bijective mapping from the set of positive semidefinite complex matrices $(c_{n,m})$ with the diagonal elements equal to one to the set of the classes $\{E^\alpha \mid \alpha \in \mathbb{R}\}$ of α-shifted phase observables where any E^α in the same class have the same phase matrix. In the rest of this paper we indentify the α-shifted phase observables which have the same phase matrix and briefly call the α-shifted phase observables as phase observables. We also drop out the index α from the symbol E^α.

2.2. *Generalized operator measures*

One may ask if the converse statement of theorem 2.1 is true. We study this question next.

Let $(c_{n,m})_{n,m\in\mathbb{N}}$ be an infinite-dimensional complex matrix and suppose that $c_{n,n} \equiv 1$. Let $\mathcal{M} := \mathrm{lin}\{|n\rangle \mid n \in \mathbb{N}\}$ and define the following function for all $\varphi, \psi \in \mathcal{M}$:

$$\mathbb{R} \ni \theta \mapsto C_{\varphi,\psi}(\theta) := \sum_{n,m=0}^{\infty} c_{n,m} \mathrm{e}^{\mathrm{i}(n-m)\theta} \langle\varphi|n\rangle\langle m|\psi\rangle \in \mathbb{C}.$$

For a fixed $\alpha \in \mathbb{R}$, define

$$E_{\varphi,\psi}([\alpha - \pi, \alpha + \pi)) := \frac{1}{2\pi} \int_{\alpha-\pi}^{\alpha+\pi} C_{\varphi,\psi}(\theta)\, \mathrm{d}\theta = \langle\varphi|\psi\rangle.$$

Then $(\varphi, \psi) \mapsto E_{\varphi,\psi}([\alpha - \pi, \alpha + \pi))$ is a bounded sesquilinear form on $\mathcal{M}$ which is a dense linear subspace of $\mathcal{H}$. Hence it has a unique bounded extension to $\mathcal{H}$ which is $(\varphi, \psi) \mapsto \langle\varphi|\psi\rangle$. Thus, we can formally write $E([\alpha - \pi, \alpha + \pi)) = I$.

Consider the following sesquilinear form defined for all (Borel) subsets X of $[\alpha - \pi, \alpha + \pi)$:

$$\mathcal{M} \times \mathcal{M} \ni (\varphi, \psi) \mapsto E_{\varphi,\psi}(X) := \frac{1}{2\pi} \int_X C_{\varphi,\psi}(\theta)\, \mathrm{d}\theta \in \mathbb{C}.$$

The form $(\varphi, \psi) \mapsto E_{\varphi,\psi}(X)$ need not be bounded, so that it does not necessarily define a bounded operator on $\mathcal{H}$. Also it is possible that there are some $\varphi, \psi \in \mathcal{H}\backslash\mathcal{M}$ for which the mapping $\theta \mapsto \sum_{n,m=0}^{\infty} c_{n,m} \mathrm{e}^{\mathrm{i}(n-m)\theta} \langle\varphi|n\rangle\langle m|\psi\rangle$ cannot be defined for dθ-almost all $\theta \in \mathbb{R}$, and if it is defined for some $\varphi', \psi' \in \mathcal{H}\backslash\mathcal{M}$, the function $\theta \mapsto \sum_{n,m=0}^{\infty} c_{n,m} \mathrm{e}^{\mathrm{i}(n-m)\theta} \langle\varphi'|n\rangle\langle m|\psi'\rangle$ need not be integrable over $[\alpha - \pi, \alpha + \pi)$. To conclude, the formal notations

$$E(X) = \sum_{n,m=0}^{\infty} c_{n,m} \frac{1}{2\pi} \int_X \mathrm{e}^{\mathrm{i}(n-m)\theta}\, \mathrm{d}\theta |n\rangle\langle m| \tag{4}$$

$$E([\alpha - \pi, \alpha + \pi)) = I \tag{5}$$

must be understood as the sesquilinear forms defined on the largest possible domain $\mathcal{D} \subseteq \mathcal{H}$ where for all $\varphi, \psi \in \mathcal{D}$, $\sum_{n,m=0}^{\infty} c_{n,m} \mathrm{e}^{\mathrm{i}(n-m)\theta} \langle\varphi|n\rangle\langle m|\psi\rangle$ is defined for dθ-almost all $\theta \in \mathbb{R}$ and $(2\pi)^{-1} \int_0^{2\pi} \sum_{n,m=0}^{\infty} c_{n,m} \mathrm{e}^{\mathrm{i}(n-m)\theta} \langle\varphi|n\rangle\langle m|\psi\rangle\, \mathrm{d}\theta = \langle\varphi|\psi\rangle$. Note that always $\mathcal{M} \subseteq \mathcal{D}$ and $\mathcal{D}$ is a dense linear subspace of $\mathcal{H}$.

Let $\mathcal{H}_1 := \{\varphi \in \mathcal{H} \mid \sum_{n=0}^{\infty} |\langle n|\varphi\rangle| < \infty\}$ and suppose that $|c_{n,m}| \leqslant b < \infty$ for all $n, m \in \mathbb{N}$. Then $\left|\sum_{n,m=0}^{\infty} c_{n,m} \mathrm{e}^{\mathrm{i}(n-m)\theta} \langle\varphi|n\rangle\langle m|\psi\rangle\right| \leqslant b \sum_{n=0}^{\infty} |\langle n|\varphi\rangle| \sum_{m=0}^{\infty} |\langle m|\psi\rangle|$ for all $\varphi, \psi \in \mathcal{H}_1$ and $\theta \in \mathbb{R}$, so that, in this case, $\mathcal{H}_1 \subseteq \mathcal{D}$. Next we define the concept of a generalized operator measure.

Let $\mathcal{K}$ be a linear subspace of $\mathcal{H}$, and let $\mathcal{SL}(\mathcal{K}, \mathcal{K}; \mathbb{C})$ be the set of sesquilinear forms from $\mathcal{K} \times \mathcal{K}$ to $\mathbb{C}$.

Definition 2.1. *We say that a mapping $G\colon \mathcal{B}(\Omega) \to \mathcal{SL}(\mathcal{K}, \mathcal{K}; \mathbb{C})$ is a generalized operator measure if for all $\varphi,\ \psi \in \mathcal{K}$ the mapping*

$$\mathcal{B}(\Omega) \ni X \mapsto [G(X)](\varphi, \psi) \in \mathbb{C}$$

is a complex measure. If $\mathcal{K}$ is dense and $[G(\Omega)](\varphi, \psi) = \langle\varphi|\psi\rangle$ for all $\varphi,\ \psi \in \mathcal{K}$ we say that G is normalized and denote $G(\Omega) = I$. If $[G(X)](\varphi, \psi) = \overline{[G(X)](\psi, \varphi)}$, or $[G(X)](\psi, \psi) \geqslant 0$, for all $X \in \mathcal{B}(\Omega)$ and for all $\varphi,\ \psi \in \mathcal{K}$ we say that G is symmetric, or positive, respectively.

If for a normalized generalized operator measure (GOM)

$$G^{\alpha}\colon \mathcal{B}([\pi - \alpha, \pi + \alpha)) \to \mathcal{SL}(\mathcal{K}, \mathcal{K}; \mathbb{C}),$$

$\alpha \in \mathbb{R}$, the conditions $R(\theta)\mathcal{K} = \mathcal{K}$ and

$$[G^{\alpha}(X)](R(\theta)^*\varphi, R(\theta)^*\psi) = [G^{\alpha}(X \oplus \theta)](\varphi, \psi)$$

hold for all $\theta \in [0, 2\pi)$, $X \in \mathcal{B}([\pi - \alpha, \pi + \alpha))$ and $\varphi,\ \psi \in \mathcal{K}$, then G_{α} is (phase shift) covariant.

Following the proof of phase theorem 2.2 of [3] it is easy to show that if $|n\rangle,\ |m\rangle \in \mathcal{K}$ then $[G^{\alpha}(X)](|n\rangle, |m\rangle) = d_{n,m} (2\pi)^{-1} \int_X \mathrm{e}^{\mathrm{i}(n-m)\theta}\, \mathrm{d}\theta$ for all X where $d_{n,m} \in \mathbb{C}$. If $|n\rangle,\ |m\rangle \in \mathcal{K}$ for all $n,\ m \in \mathbb{N}$ we say that $(d_{n,m})_{n,m\in\mathbb{N}}$ is the *structure matrix* of a covariant GOM G^{α}. Note that $d_{n,n} = 1$ if $|n\rangle \in \mathcal{K}$.

Since $R(\theta)\mathcal{D} = \mathcal{D}$ it follows that E defined in (4) is a covariant GOM and $(c_{n,m})$ is its structure matrix. If E is symmetric or, equivalently, $c_{n,m} = \overline{c_{m,n}}$ for all $n, m \in \mathbb{N}$, we say that E is a *quasi phase observable*. The name quasi phase observable comes from the observation that for any quasi phase observable E and for a unit vector $\psi \in \mathcal{D}$ the mapping $X \mapsto E_{\psi,\psi}(X)$ is a *quasi probability measure*, that is, a normalized real measure. An example of a quasi probability measure is the polar coordinate margin measure of the Wigner function of a state (when it exists). Indeed, as will be shown in section 5.2, this measure is related to a certain quasi phase observable.

Finally, if E is positive then the matrix $(c_{n,m})$ is positive semidefinite. Hence, $0 \leqslant E_{\varphi,\varphi}(X) \leqslant E_{\varphi,\varphi}([\alpha-\pi,\alpha+\pi)) = \|\varphi\|^2$ and $(\varphi,\psi) \mapsto E_{\varphi,\psi}(X)$ is bounded on $\mathcal{M}$. In this case, $\mathcal{D} = \mathcal{H}$ and the sesquilinear form $E(X)$ can be regarded as a bounded operator with the unique matrix elements $E_{|n\rangle,|m\rangle}(X) = c_{n,m}(2\pi)^{-1}\int_X \mathrm{e}^{\mathrm{i}(n-m)\theta}\,\mathrm{d}\theta$, $n, m \in \mathbb{N}$. The mapping $E: \mathcal{B}([\alpha-\pi,\alpha+\pi)) \to \mathcal{L}(\mathcal{H})$ is σ-additive (see the proof of phase theorem 2.2 of [3]). Thus, equation (4) defines an (α-shifted) phase observable $X \mapsto E(X)$.

3. Phase operators

For any $\alpha \in \mathbb{R}$ let $E: \mathcal{B}([\alpha-\pi,\alpha+\pi)) \to \mathcal{L}(\mathcal{H})$ be a covariant normalized (not necessarily positive) operator measure, and let $E_{\psi,\varphi}$ denote the complex measure $X \mapsto E_{\psi,\varphi}(X) := \langle\psi|E(X)\varphi\rangle$. The *first moment operator* of E is defined as

$$\langle\psi|\Phi\varphi\rangle := \int_{\alpha-\pi}^{\alpha+\pi} \theta\,\mathrm{d}E_{\psi,\varphi}(\theta) \qquad \psi,\varphi \in \mathcal{H}.$$

The operator Φ is bounded, and if E is self-adjoint then Φ is self-adjoint. From theorem 2.1 one gets

$$\Phi = \alpha I + \sum_{n\neq m=0}^{\infty} c_{n,m}\frac{\mathrm{i}}{m-n}\mathrm{e}^{\mathrm{i}(n-m)(\alpha-\pi)}|n\rangle\langle m| \tag{6}$$

which implies that

$$c_{n,m} = \Phi_{n,m}\mathrm{i}(n-m)\mathrm{e}^{\mathrm{i}(n-m)(\pi-\alpha)} \qquad n \neq m. \tag{7}$$

Let A be a bounded operator for which $A_{n,n} = \alpha$ for all $n \in \mathbb{N}$ and for some $\alpha \in \mathbb{R}$. Define $a_{n,n} := 1$ and $a_{n,m} := A_{n,m}\mathrm{i}(n-m)\mathrm{e}^{\mathrm{i}(n-m)(\pi-\alpha)}$ for all $n \neq m$. If the matrix $(a_{n,m})_{n,m\in\mathbb{N}}$ is positive semidefinite then $X \mapsto \sum_{n,m=0}^{\infty} a_{n,m}(2\pi)^{-1}\int_X \mathrm{e}^{\mathrm{i}(n-m)\theta}\,\mathrm{d}\theta|n\rangle\langle m|$ is an (α-shifted) phase observable whose first moment operator is A. This shows that already the first moment operator Φ of a phase observable E determines E uniquely. Hence, if E is a phase observable we say that Φ is the *phase operator* associated with the phase observable E.

Remark 3.1. Let $(c_{n,m})_{n,m\in\mathbb{N}}$ be an infinite-dimensional complex matrix with $c_{n,n} \equiv 1$ and fix $\alpha \in \mathbb{R}$. For all $X \in \mathcal{B}([\alpha-\pi,\alpha+\pi))$ let $E(X)$ be the sesquilinear form defined on $\mathcal{D}$ (see equation (4)). Integrating the identity function with respect to the complex measure $X \mapsto E_{\varphi,\psi}(X)$, $\varphi, \psi \in \mathcal{M}$, yields the following sesquilinear form, *the first moment form:*

$$(\varphi,\psi) \mapsto \Phi_{\varphi,\psi} := \int_{\alpha-\pi}^{\alpha+\pi} \theta\,\mathrm{d}E_{\varphi,\psi}(\theta) = \sum_{n,m=0}^{\infty} c_{n,m}\frac{1}{2\pi}\int_{\alpha-\pi}^{\alpha+\pi} \theta\mathrm{e}^{\mathrm{i}(n-m)\theta}\,\mathrm{d}\theta\langle\varphi|n\rangle\langle m|\psi\rangle.$$

If it is bounded then $(\varphi,\psi) \mapsto \Phi_{\varphi,\psi}$ defines the bounded operator

$$\Phi := \alpha I + \sum_{n\neq m=0}^{\infty} c_{n,m}\frac{\mathrm{i}}{m-n}\mathrm{e}^{\mathrm{i}(n-m)(\alpha-\pi)}|n\rangle\langle m| \tag{8}$$

and we say that Φ is the *first moment operator* of the covariant GOM E. If $c_{n,m} = \overline{c_{m,n}}$, $n, m \in \mathbb{N}$, then Φ is self-adjoint. Note that any bounded operator A, $A_{n,n} \equiv \alpha$, is the first moment operator of a unique covariant GOM based on $[\alpha-\pi,\alpha+\pi)$ which has the matrix $(a_{n,m})_{n,m\in\mathbb{N}}$, $a_{n,n} \equiv 1$, $a_{n,m} \equiv A_{n,m}\mathrm{i}(n-m)\mathrm{e}^{\mathrm{i}(n-m)(\pi-\alpha)}$, as its structure matrix.

Remark 3.2. Let E be a covariant GOM based on $[\alpha-\pi,\alpha+\pi)$ and $(c_{n,m})$ be its structure matrix. Let $\varphi, \psi \in \mathcal{M}$. It is easy to see that

$$\Phi_{N\varphi,\psi} - \Phi_{\varphi,N\psi} = \mathrm{i}[\langle\varphi|\psi\rangle - C_{\varphi,\psi}(\alpha-\pi)]. \tag{9}$$

Since for all $\varphi, \psi \in \mathcal{M}$

$$\int_{\alpha-\pi}^{\alpha+\pi} 2\pi\delta_{2\pi}(\theta-\alpha+\pi)\,\mathrm{d}E_{\varphi,\psi}(\theta) = C_{\varphi,\psi}(\alpha-\pi)$$

where $\delta_{2\pi}(\theta)$ is a 2π-periodic Dirac delta distribution, the expression (9) corresponds to the classical Poisson bracket

$$\{H,\phi\}_{\mathrm{PB}} = 1 - 2\pi\delta_{2\pi}(\theta-\alpha+\pi) \tag{10}$$

where H is the classical oscillator energy and $\phi \in [\alpha-\pi, \alpha+\pi)$ is a single-valued classical phase (see section II of [4]).

Remark 3.3. The phase operator Φ of a phase observable E based on $[\alpha-\pi, \alpha+\pi)$ is self-adjoint and bounded with

$$(\alpha-\pi)I \leqslant \Phi \leqslant (\alpha+\pi)I.$$

Let F be the unique spectral measure of Φ. Since no phase observable is projection valued it follows that $E \neq F$.

The conventional interpretation of quantum mechanics identifies self-adjoint operators and observables. One might thus consider the spectral measure F of Φ as a phase observable. However, there are several reasons why F cannot describe a quantum phase observable:

(1) F does not obey the phase shift covariance condition which is essential for the interpretation of phase measurement statistics. Although the first moment operators of E and F coincide, $\int\theta\,\mathrm{d}E(\theta) = \int\theta\,\mathrm{d}F(\theta)$, the other moments are not the same. For example, $\int\theta^2\,\mathrm{d}E(\theta)$ is strictly greater than $(\int\theta\,\mathrm{d}E(\theta))^2 = \int\theta^2\,\mathrm{d}F(\theta)$ (see e.g. [12, appendix, section 3]). This means, in particular, that the moments of a phase probability measure $X \mapsto \mathrm{tr}(E(X)T)$ related to a measurement of the phase observable E in a state T (which is a positive trace-one operator) are not the moments of the probability measure $X \mapsto \mathrm{tr}(F(X)T)$.

(2) The support of E is always the interval $[\alpha-\pi, \alpha+\pi]$ whereas in many cases the spectrum of Φ (or the support of F) is a proper subset of $[\alpha-\pi, \alpha+\pi]$. For example, in the case of the trivial phase observable $E_{\mathrm{triv}}(X) := (2\pi)^{-1}\int_X \mathrm{d}\theta\, I$, $X \in \mathcal{B}([0, 2\pi))$, the probability measure $X \mapsto \mathrm{tr}(TE_{\mathrm{triv}}(X))$ in a state T is uniformly distributed on $[0, 2\pi)$ and, thus, describes a trivial phase measurement where one cannot get any information, for example, from the phase of a large amplitude coherent state. The support of the spectral measure F_{triv} of the phase operator $\Phi_{\mathrm{triv}} = \pi I$ is the one-point set $\{\pi\}$, so that it is hard to consider F_{triv} as a phase observable.

(3) Using the fact that $|c_{n,m}| \leqslant 1$, $n, m \in \mathbb{N}$, for the phase matrix elements of E, one easily calculates that

$$\langle n|\Phi^2|n\rangle - (\langle n|\Phi|n\rangle)^2 = \sum_{\substack{m=0\\ m\neq n}}^{\infty} \frac{|c_{n,m}|^2}{(n-m)^2} < \frac{\pi^2}{3}$$

which shows that F is *never* random in number states. On the other hand, E *always* gives the random phase distribution in number states.

Finally, note that there are realistic measurement schemes constructed for the measurements of different phase observables. In particular, this applies to all the so-called phase space phase observables $E^{|n\rangle}$, $n \in \mathbb{N}$ [13]. Actually $E^{|0\rangle}$ (or the angle margin of the Husimi Q-function) has already been measured in some states using a double homodyne detection.

Example 3.1. Consider the first moment operator of the canonical phase $E_{\rm can}$ (or the Garrison–Wong–Galindo operator [14–16])

$$\Phi_{\rm can} = \pi I + \sum_{n\neq m=0}^{\infty} \frac{\mathrm{i}}{m-n} |n\rangle\langle m|. \tag{11}$$

Its spectral measure $F_{\rm can}$ as well as $E_{\rm can}$ has the support $[0, 2\pi]$, and they both behave well in the classical limit of coherent states [11, 14, 17]. However, only $E_{\rm can}$ is covariant, and thus, describes a true phase observable.

Remark 3.4. Let A be a bounded self-adjoint operator with $A_{n,n} \equiv \alpha$ and E be its (unique) quasi phase observable based on $[\alpha - \pi, \alpha + \pi)$. Let F be a spectral measure of A. The situation differs from the case of a phase operator since E may not be positive. Hence, the quasi probability measure $X \mapsto E_{\psi,\psi}(X)$, $\psi \in \mathcal{D}$, $\|\psi\| = 1$, may get negative values and, thus, it cannot be considered as a probability measure. Still the measure $X \mapsto F_{\psi,\psi}(X)$ is a probability measure.

As we shall see in section 5.2 the symmetrically ordered phase operator (the Wigner–Weyl quantized phase angle) is determined by a quasi phase observable which is not a POM. Its spectral measure has been suggested as a phase observable (although it is not covariant), since its support seems to be the whole interval $[0, 2\pi]$ and it behaves well in the classical limit of coherent states [6–9]. The spectral measure of the symmetrically ordered phase operator does not give a random phase distribution in number states [6, p. 458].

4. Covariant angle margins of the phase space operator measures

Let $D(z) := \mathrm{e}^{za^*-\bar{z}a}$, $z \in \mathbb{C}$, be a unitary shift operator and ν: $\mathcal{B}(\mathbb{C}) \to [0, \infty]$ be the two-dimensional Lebesgue measure. For any trace class operator $\Delta \in \mathcal{L}(\mathcal{H})$

$$\frac{1}{\pi}\int_{\mathbb{C}} D(z)\Delta D(z)^* \,\mathrm{d}\nu(z) = \mathrm{tr}(\Delta) I \tag{12}$$

holds (see e.g. [18]). If $\mathrm{tr}(\Delta) = 1$ then one can define the following normalized operator measure:

$$\mathcal{B}(\mathbb{C}) \ni Z \mapsto A^{\Delta}(Z) := \frac{1}{\pi}\int_{Z} D(z)\Delta D(z)^* \,\mathrm{d}\nu(z) \in \mathcal{L}(\mathcal{H}) \tag{13}$$

with the associated operator density

$$\mathbb{C} \ni z \mapsto D(z)\Delta D(z)^* \in \mathcal{L}(\mathcal{H}). \tag{14}$$

We say that A^{Δ} is an *operator ordering measure* or a *phase space operator measure* and Δ is the *generator* of A^{Δ}. If $\Delta = T$ is a state then A^T is positive, and it is called a *phase space observable*.

Writing $\mathbb{C} \ni z = |z|\mathrm{e}^{\mathrm{i}\arg z} \equiv r\mathrm{e}^{\mathrm{i}\theta}$, $(r, \theta) \in [0, \infty) \times [0, 2\pi)$, we may define the polar coordinate angle margin measure of A^{Δ}:

$$\mathcal{B}([0, 2\pi)) \ni X \mapsto E^{\Delta}(X) := A^{\Delta}([0, \infty) \times X) \in \mathcal{L}(\mathcal{H}). \tag{15}$$

If E^{Δ} is covariant (under phase shifts) then A^{Δ} is so-called *covariant operator ordering measure* [4]. If the angle margin E^T of a phase space observable A^T is covariant we say that E^T is a *phase space phase observable*. Theorem 4.1 of [3] states that for any state T the angle margin E^T of a phase space observable A^T is a phase observable if and only if T is of the form

$$T = \sum_{k=0}^{\infty} \lambda_k |k\rangle\langle k|$$

where $\lambda_k \geqslant 0, k \in \mathbb{N}$ and $\sum_{k=0}^{\infty} \lambda_k = 1$. Especially, when $T = |k\rangle\langle k|$, $k \in \mathbb{N}$, we write $A^T \equiv A^{|k\rangle}$ and $E^T \equiv E^{|k\rangle}$. Using theorem 2.1 and the proof of theorem 4.1 of [3] one can prove the following generalization:

Theorem 4.1. *Let Δ be a trace-one operator. Then the angle margin E^{Δ} of a phase space operator measure A^{Δ} is covariant if and only if the generator Δ of A^{Δ} is of the form*

$$\Delta = \sum_{k=0}^{\infty} \lambda_k |k\rangle\langle k|$$

where $\lambda_k \in \mathbb{C}$, $k \in \mathbb{N}$, $\sum_{k=0}^{\infty} |\lambda_k| < \infty$ and $\sum_{k=0}^{\infty} \lambda_k = 1$.

In the context of theorem 4.1 one may write

$$A^{\Delta} = \sum_{k=0}^{\infty} \lambda_k A^{|k\rangle} \tag{16}$$

$$E^{\Delta} = \sum_{k=0}^{\infty} \lambda_k E^{|k\rangle} \tag{17}$$

$$c^{\Delta}_{n,m} = \sum_{k=0}^{\infty} \lambda_k c^{|k\rangle}_{n,m} \qquad n, m \in \mathbb{N} \tag{18}$$

(with the convergence in the weak operator topology) where $(c^{|k\rangle}_{n,m})$ is the structure matrix of $E^{|k\rangle}$. From [3] one gets for all $n, m, k \in \mathbb{N}$ that

$$c^{|k\rangle}_{n,m} = (-1)^{\max\{0,k-n\}+\max\{0,k-m\}} \sqrt{\frac{(\min\{n,k\})!(\min\{m,k\})!}{(\max\{n,k\})!(\max\{m,k\})!}} \times \int_0^{\infty} \mathrm{e}^{-x} x^{(|k-n|+|k-m|)/2} L^{|k-n|}_{\min\{n,k\}}(x) L^{|k-m|}_{\min\{m,k\}}(x)\,\mathrm{d}x \tag{19}$$

where L^{α}_l is the associated Laguerre polynomial.

5. Cahill–Glauber *s*-ordered phase operators

Let $\lambda \in \mathbb{C}$, $r \in [0, \infty)$, $\theta \in [0, 2\pi)$ and $n, m, k \in \mathbb{N}$. One can show (see e.g. [10]) that when $\lambda \neq 0$

$$\sum_{k=0}^{\infty} \lambda^k D(r)_{n,k} D(r)_{m,k} = \sqrt{\frac{(\min\{n,m\})!}{(\max\{n,m\})!}} (1-\lambda)^{|n-m|} \lambda^{\min\{n,m\}} \times \mathrm{e}^{(\lambda-1)r^2} r^{|n-m|} L^{|n-m|}_{\min\{n,m\}}((2-\lambda-\lambda^{-1})r^2)$$

and

$$\begin{aligned}
&\int_0^{2\pi} \frac{1}{2\pi} \int_0^{\infty} (1-\lambda) \sum_{k=0}^{\infty} \lambda^k D(r\mathrm{e}^{\mathrm{i}\theta})_{n,k} D(r\mathrm{e}^{\mathrm{i}\theta})^*_{k,m}\,\mathrm{d}r^2\,\mathrm{d}\theta \\
&\quad = \begin{cases} \delta_{n,m}(1-\lambda)\lambda^n \int_0^{\infty} \mathrm{e}^{(\lambda-1)r^2} L^0_n((2-\lambda-\lambda^{-1})r^2)\,\mathrm{d}r^2 & \text{when } \lambda \neq 0 \\ \delta_{n,m} \frac{1}{n!} \int_0^{\infty} \mathrm{e}^{-r^2} r^{2n}\,\mathrm{d}r^2 & \text{when } \lambda = 0 \end{cases} \\
&\quad = \delta_{n,m} \qquad \text{if and only if } \operatorname{Re}\lambda < 1.
\end{aligned}$$

Suppose that $\mathrm{Re}\,\lambda < 1$ and let $\mathcal{D}^\lambda$ consists of those vectors φ, ψ for which the sequence

$$p \mapsto \sum_{k=0}^{p} \lambda^k \langle\varphi|D(z)|k\rangle\langle k|D(z)^*|\psi\rangle$$

converges for ν-almost all $z \in \mathbb{C}$ and

$$\frac{1}{\pi}\int_{\mathbb{C}} (1-\lambda)\sum_{k=0}^{\infty} \lambda^k \langle\varphi|D(z)|k\rangle\langle k|D(z)^*|\psi\rangle \,\mathrm{d}\nu(z) = \langle\varphi|\psi\rangle.$$

Now $\mathcal{M} \subseteq \mathcal{D}^\lambda$ and $\mathcal{D}^\lambda$ is dense linear subspace of $\mathcal{H}$. Also the coherent states $|z\rangle := D(z)|0\rangle$, $z \in \mathbb{C}$, are elements of $\mathcal{D}^\lambda$. Thus, one can define for all (Borel) subsets Z of $\mathbb{C}$ a sesquilinear form on $\mathcal{D}^\lambda$ as

$$(\varphi, \psi) \mapsto \tilde{A}^\lambda_{\varphi,\psi}(Z) := \frac{1}{\pi}\int_Z (1-\lambda)\sum_{k=0}^{\infty} \lambda^k \langle\varphi|D(z)|k\rangle\langle k|D(z)^*|\psi\rangle \,\mathrm{d}\nu(z).$$

Since $(\varphi,\psi) \mapsto \tilde{A}^\lambda_{\varphi,\psi}(\mathbb{C}) = \langle\varphi|\psi\rangle$ is a bounded sesquilinear form defined on the dense linear subspace, it has a unique extension to $\mathcal{H}$, namely $(\varphi,\psi) \mapsto \langle\varphi|\psi\rangle$. Thus for all $\mathrm{Re}\ \lambda < 1$ we may formally write

$$\tilde{A}^\lambda(Z) = \frac{1}{\pi}\int_Z D(z)(1-\lambda)\lambda^N D(z)^* \,\mathrm{d}\nu(z) \tag{20}$$

$$\tilde{A}^\lambda(\mathbb{C}) = \frac{1}{\pi}\int_{\mathbb{C}} D(z)(1-\lambda)\lambda^N D(z)^* \,\mathrm{d}\nu(z) = I. \tag{21}$$

The operator

$$(1-\lambda)\lambda^N = (1-\lambda)\sum_{k=0}^{\infty} \lambda^k |k\rangle\langle k| \tag{22}$$

is bounded when $|\lambda| \leqslant 1$. If $|\lambda| < 1$ then it is trace class operator, $\mathcal{D}^\lambda = \mathcal{H}$, and $\tilde{A}^\lambda : \mathcal{B}(\mathbb{C}) \to \mathcal{L}(\mathcal{H})$ is a phase space operator measure. If $\mathcal{D}^\lambda \neq \mathcal{H}$, $\tilde{A}^\lambda$, must be regarded as a sesquilinear form valued mapping, that is, a GOM.

Gahill and Glauber [10] defined an *s-ordered displacement operator* $D^s(z) := \mathrm{e}^{s|z|^2/2}D(z)$ for all $s, z \in \mathbb{C}$. By direct calculation one gets

$$\frac{1}{\pi}\int_{\mathbb{C}} \langle n|D^s(z)\,|m\rangle \,\mathrm{d}\nu(z) = \delta_{n,m}\frac{2}{1-s}\left(\frac{s+1}{s-1}\right)^n$$

for all $n, m \in \mathbb{N}$ when $\mathrm{Re}\,s < 1$. If $\mathrm{Re}\,s \geqslant 1$, then the integral does not exist. Thus, when $\mathrm{Re}\,s < 1$

$$\Delta_s := \frac{1}{\pi}\int_{\mathbb{C}} D^s(z)\,\mathrm{d}\nu(z) = (1-\lambda)\lambda^N \tag{23}$$

where $\lambda = (s+1)/(s-1)$, that is, $s = (\lambda+1)/(\lambda-1)$. The mapping $s \mapsto \frac{s+1}{s-1}$ is bijective on $\mathbb{C}\backslash\{1\}$ and the condition $\mathrm{Re}\,s < 1$ is equivalent to $\mathrm{Re}\,\lambda < 1$. In the rest of this paper, we assume that the parameters λ and s are related to each others in the above way, and we also write $A^s := \tilde{A}^\lambda$ and $\mathcal{D}^s := \mathcal{D}^\lambda$ for all $\mathrm{Re}\,s < 1$

Using equation (20) we can write formally for all $\mathrm{Re}\,s < 1$ that

$$A^s(Z) = \frac{1}{\pi}\int_Z D(z)\Delta_s D(z)^* \,\mathrm{d}\nu(z) = \frac{1}{\pi}\int_Z \frac{1}{\pi}\int_{\mathbb{C}} \mathrm{e}^{z\bar{\xi}-\bar{z}\xi} D^s(\xi)\,\mathrm{d}\nu(\xi)\,\mathrm{d}\nu(z) \tag{24}$$

where $\pi^{-1}\int_{\mathbb{C}} \mathrm{e}^{z\bar{\xi}-\bar{z}\xi} D^s(\xi)\,\mathrm{d}\nu(\xi) = D(z)\Delta_s D(z)^*$ is a (possibly unbounded) Fourier transformed s-ordered displacement operator [10].

5.1. Phase observables

Suppose that $\operatorname{Re} s < 1$ and $\varphi, \psi \in \mathcal{D}^s$. Define the angle margin of the measure $A^s_{\varphi,\psi}$ as

$$\mathcal{B}([0, 2\pi)) \ni X \mapsto E^s_{\varphi,\psi}(X) := A^s_{\varphi,\psi}([0,\infty) \times X) \in \mathbb{C} \tag{25}$$

which determines a GOM E^s. Since $R(\theta)\mathcal{D}^s = \mathcal{D}^s$ it follows that E^s is covariant GOM (see section 2.2). Thus, if $\operatorname{Re} s < 1$ one gets $E^s_{|n\rangle,|m\rangle}(X) = c^s_{n,m}(2\pi)^{-1}\int_X \mathrm{e}^{\mathrm{i}(n-m)\theta}\,\mathrm{d}\theta$ where

$$c^s_{n,m} = (1-\lambda)\int_0^\infty \sum_{k=0}^\infty \lambda^k D(r)_{n,k} D(r)_{m,k}\,\mathrm{d}r^2 = \sqrt{\frac{(\min\{n,m\})!}{(\max\{n,m\})!}}(1-\lambda)^{|n-m|+1}\lambda^{\min\{n,m\}}$$
$$\times \int_0^\infty \mathrm{e}^{(\lambda-1)r^2} r^{|n-m|} L^{|n-m|}_{\min\{n,m\}}((2-\lambda-\lambda^{-1})r^2)\,\mathrm{d}r^2 \tag{26}$$

when $\lambda \neq 0$ or $s \neq -1$ (see also [10], [19] or [4]). If $s = -1$ then $c^{s=-1}_{n,m} = \int_0^\infty D(r)_{n,0} D(r)_{m,0}\,\mathrm{d}r^2 = c^{|0\rangle}_{n,m}$ and

$$\Delta^{s=-1} = |0\rangle\langle 0| \qquad A^{s=-1} = A^{|0\rangle} \qquad E^{s=-1} = E^{|0\rangle}. \tag{27}$$

From equation (26) one sees that

$$c^s_{2n,0} = (1-\lambda)^n \frac{n!}{\sqrt{(2n)!}} \sim \sqrt[4]{\pi}\left(\frac{1-\lambda}{2}\right)^n n^{1/4} \tag{28}$$

when $n \to \infty$, and

$$c^s_{2,0} = \frac{1-\lambda}{\sqrt{2}} = c^s_{0,2}.$$

Thus, $c^s_{2,0} = \overline{c^s_{0,2}}$ if and only if $\lambda \in \mathbb{R}$. This implies that

Proposition 5.1. *A^s is a symmetric GOM and E^s is a quasi phase observable if and only if $\lambda \in \mathbb{R}$ and $\lambda < 1$, or $s \in \mathbb{R}$ and $s < 1$.*

We also see that

(1) if $|\lambda| < 1$, or $\operatorname{Re} s < 0$, then Δ_s is a trace-class operator with $\operatorname{tr}(\Delta_s) = 1$ and $\|\Delta_s\| = |1-\lambda|$, and it satisfies the conditions of theorem 4.1. Hence we can write

$$A^s(Z) = (1-\lambda)\sum_{k=0}^\infty \lambda^k A^{|k\rangle}(Z) \tag{29}$$

$$E^s(X) = (1-\lambda)\sum_{k=0}^\infty \lambda^k E^{|k\rangle}(X) \tag{30}$$

$$c^s_{n,m} = (1-\lambda)\sum_{k=0}^\infty \lambda^k c^{|k\rangle}_{n,m}. \tag{31}$$

Moreover, A^s has an operator density $z \mapsto D(z)\Delta_s D(z)^*$.

(2) If $|\lambda| = 1$ and $\lambda \neq 1$, or $\operatorname{Re} s = 0$, then we can write $\lambda = \mathrm{e}^{\mathrm{i}\gamma}$ where $\gamma = \arg\lambda$ and $\gamma \neq 0$. Now $\Delta_s = (1-\mathrm{e}^{\mathrm{i}\gamma})R(\gamma)$ is not a trace-class operator but it is bounded with $\|\Delta_s\| = |1-\mathrm{e}^{\mathrm{i}\gamma}|$. Although we can define a bounded operator valued function

$$z \mapsto D(z)\Delta_s D(z)^* = (1-\mathrm{e}^{\mathrm{i}\gamma})\mathrm{e}^{\mathrm{i}|z|^2\sin\gamma} D(z(1-\mathrm{e}^{\mathrm{i}\gamma}))R(\gamma)$$

it does not follow that A^s and E^s are operator measures. Indeed, if one chooses $\gamma = \pi$ then $z \mapsto \operatorname{tr}(TD(z)\Delta_{s=0}D(z)^*) = 2\operatorname{tr}(TD(2z)R(\pi))$ is the Wigner function of a state T which is not integrable over $\mathbb{C}$ for all pure states $|\psi\rangle\langle\psi|$ (see e.g. [20]). Thus, $\mathcal{D}^{s=0}$ is a proper subset of $\mathcal{H}$. Finally, it can be seen directly that equations (29)–(31) cannot be true (take $Z = \mathbb{C}$, $X = [0, 2\pi)$, and $n = m$ in equations (29)–(31)).

(3) If $|\lambda| > 1$ and $\operatorname{Re}\lambda < 1$, or $0 < \operatorname{Re}s < 1$, then Δ_s is unbounded and equation (20) must be seen as a formal notation. Obviously A^s has no operator density. From equation (28) one sees that

$$\sup_{n\in\mathbb{N}}\left\{\left|E^s_{|n\rangle,|0\rangle}([0,\pi/2))\right|\right\} = \infty$$

when $\lambda \in \mathbb{R}$, $\lambda < -1$, or $s \in \mathbb{R}$, $0 < s < 1$, which implies that E^s and A^s are not operator measures when $0 < s < 1$.

(4) If $\operatorname{Re}\lambda \geqslant 1$, or $\operatorname{Re}s \geqslant 1$, then A^s cannot be defined at all.

Suppose that $\lambda < 1$. Now $c^s_{0,1} = \sqrt{\pi}/2\sqrt{1-\lambda} > 1$ when $\lambda < 1 - 4/\pi \approx -0.27$. Hence, when $\lambda < 1 - 4/\pi$ it follows that E^s, and thus A^s, cannot be positive. If $-1 < \lambda < 1$ then $\langle\varphi|A^s(Z)\psi\rangle = \pi^{-1}\int_Z\langle\varphi|D(z)\Delta_s D(z)^*\psi\rangle\,\mathrm{d}\nu(z)$ for all $Z \in \mathcal{B}(\mathbb{C})$ and $\varphi, \psi \in \mathcal{H}$, where $z \mapsto \langle\varphi|D(z)\Delta_s D(z)^*\psi\rangle$ is a continuous function [18]. From this one sees that the positivity of A^s is equivalent to the positivity of Δ_s. The following statement is thus proved:

Proposition 5.2. *A^s is positive if and only if $0 \leqslant \lambda < 1$, or $s \leqslant -1$. When $0 \leqslant \lambda < 1$ then E^s is a phase observable.*

In the case $1 - 4/\pi \leqslant \lambda < 0$ the positivity of E^s remains an open question, since although A^s is not positive when $1 - 4/\pi \leqslant \lambda < 0$, the angle margin E^s might be positive. Numerical calculations however suggest that in this case E^s is not positive.

5.2. Phase operators

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a Borel measurable function, that is, a *dynamical variable* on the two-dimensional phase space $\mathbb{R}^2$, which we identify in the usual way with $\mathbb{C}$. One way to quantize f, that is, to find a self-adjoint operator associated with f, is to replace f by the (possibly unbounded) operator generated by the sesquilinear form $\int_{\mathbb{C}} f(z)\,\mathrm{d}A(z)$, where A is a GOM. In this context, important GOMs are the Cahill–Clauber s-parametrized GOMs A^s, $s \in \mathbb{R}$, $s < 1$, and the phase space observables $A^{|k\rangle}$, $k \in \mathbb{N}$.

Let us consider the quantization of the classical oscillator energy $|z|^2/2$. If $s = -1$ then $A^{s=-1} = A^{|0\rangle}$, and $\pi^{-1}\int_{\mathbb{C}} z\bar{z}\,\mathrm{d}A^{|0\rangle}(z) = aa^* = N + I$ [1]. The phase space observable quantizes $z\bar{z}/2$ *antinormally*. Generally, for any phase space observables one gets $\pi^{-1}\int_{\mathbb{C}} z\bar{z}\,\mathrm{d}A^{|k\rangle}(z) = aa^* + kI = N + (k+1)I$ for all $k \in \mathbb{N}$ [21]. If $s \in \mathbb{C}$ and $\operatorname{Re}s < 0$ then

$$\frac{1}{\pi}\int_{\mathbb{C}} z\bar{z}\,\mathrm{d}A^s(z) = (1-\lambda)\sum_{k=0}^{\infty}\lambda^k(aa^* + kI) = N + \frac{1}{1-\lambda}I. \tag{32}$$

The GOM corresponding to a *symmetric ordering* is $A^{s=0}$ [10]. In this case $\lambda = -1$ and $\Delta^{s=0} = 2R(\pi)$. As suggested in [10] now $\pi^{-1}\int_{\mathbb{C}} z\bar{z}D(z)2R(\pi)D(z)^*\,\mathrm{d}\nu(z) = (a^*a + aa^*)/2$, so that formally equation (32) seems to hold in this case. Also, formally, when $s \to 1-$, or $\lambda \to -\infty$, $\pi^{-1}\int_{\mathbb{C}} z\bar{z}D(z)\Delta^s D(z)^*\,\mathrm{d}\nu(z) \to a^*a$, that is, 'the operator integral in the limit $s \to 1$' seems to correspond to a *normal ordering* (see [10]).

For any GOM A^s, $s < 1$, and a vector state $\psi \in \mathcal{D}^s$, $\|\psi\| = 1$, there exists a quasiprobability density function $g^s_\psi\colon \mathbb{C} \to \mathbb{R}$ such as $A^s_{\psi,\psi}(Z) = \pi^{-1}\int_Z g^s_\psi(z)\,\mathrm{d}\nu(z)$ for all $Z \in \mathcal{B}(\mathbb{C})$. When $s = -1$, the function $g^{s=-1}_\psi(z) \equiv Q(z) = |\langle z|\psi\rangle|^2$ is the Husimi Q-function of a state ψ. If $s = 0$, then $g^{s=0}_\psi(z) \equiv W(z) = 2\langle\psi|D(z)R(\pi)D(z)^*\psi\rangle$ is the Wigner function of a state ψ.

Finally, let $\psi \in \mathcal{D}^s$ for all $\lambda < 1$. Then one may define the Glauber–Sudarshan P-distribution of a state ψ as follows:

$$P(z) := \lim_{\lambda\to-\infty}(1-\lambda)\sum_{k=0}^{\infty}\lambda^k\langle\psi|D(z)|k\rangle\langle k|D(z)^*|\psi\rangle$$

whenever the limit exists, for example, as a Dirac δ-distribution.

A natural way to quantize a classical phase $\theta = \arg z$ is to integrate θ with respect to a symmetric GOM defined on $\mathcal{B}(\mathbb{C})$. Using $A^{|n\rangle}$, $n \in \mathbb{N}$, as an operator measure one gets the *phase space phase operator* $\Phi^{|n\rangle}$.

Let $s<1$ and define a sesquilinear form $\mathcal{M}\times\mathcal{M}\ni(\varphi,\psi)\mapsto\Phi^s_{\varphi,\psi} := (2\pi)^{-1}\int_0^{2\pi}\theta\,\mathrm{d}E^s_{\varphi,\psi}(\theta) = \pi^{-1}\int_{\mathbb{C}}\arg z\,\mathrm{d}A^s_{\varphi,\psi}(z)\in\mathbb{C}$. Now

$$\Phi^s_{|n\rangle,|m\rangle} = \begin{cases} c^s_{n,m}\frac{\mathrm{i}}{m-n} & \text{when } n\neq m\\ \pi & \text{when } n = m.\end{cases}$$

If Φ^s is bounded on $\mathcal{M}\times\mathcal{M}$, then Φ^s has a unique bounded extension to $\mathcal{H}$ which defines a self-adjoint operator on $\mathcal{H}$ (see remark 3.1). In this case we denote this self-adjoint operator also by Φ^s and say that Φ^s is an *s-ordered phase operator* or an *s-quantized phase angle*.

For $s=-1$, $\Phi^{s=-1}$ is the *antinormally ordered phase operator*, and it equals the phase operator $\Phi^{|0\rangle}$. The corresponding phase observable is $E^{|0\rangle}$. If $s<0$, then $\Phi^s = (1-\lambda)\sum_{k=0}^{\infty}\lambda^k\Phi^{|n\rangle}$. When $s\leqslant -1$ then Φ^s is a phase operator determined by the phase observable E^s, and if $-1<s<0$ then Φ^s is a bounded self-adjoint operator. In the latter case, E^s seems to be a non-positive quasi phase observable.

For $s=0$ one gets the self-adjoint bounded *symmetrically ordered phase operator* or the *Wigner–Weyl quantized phase angle* $\Phi^{s=0}$ [6–9]. This operator is determined by the quasi phase observable $E^{s=0}$, which is not positive since $c^{s=0}_{0,2} = \sqrt{2} > 1$. Also Dubin and Hennings [22] have observed that $\Phi^{s=0}$ cannot be a first moment operator of a covariant POM. It is easy to show that

$$c^{s=0}_{n,m} = 2^{(n+m)/2}\sqrt{n!m!}\sum_{t=0}^{\min\{n,m\}}\left(-\frac{1}{2}\right)^t\frac{\Gamma((n+m)/2-t+1)}{t!(n-t)!(m-t)!}$$

for all $n, m \in \mathbb{N}$ (these matrix elements have also been calculated in [4, 7, 19, 20, 23]).

If $0<s<1$, or $\lambda<-1$, then using equation (28) one gets

$$|\Phi^s_{|2n\rangle,|0\rangle}| = |1-\lambda|^n\frac{n!}{2n\sqrt{(2n)!}} \sim \frac{\sqrt[4]{\pi}}{2}\left(\frac{|1-\lambda|}{2}\right)^n n^{-3/4}\to\infty$$

when $n\to\infty$. Thus, Φ^s cannot define a bounded operator.

6. Covariant GOMs in coherent states

Let E: $\mathcal{B}([\alpha-\pi,\alpha+\pi))\to\mathcal{L}(\mathcal{H})$, $\alpha\in\mathbb{R}$, be a covariant GOM and $(c_{n,m})$ be the structure matrix of E. Assume that $|z\rangle\in\mathcal{D}$ for all $z\in\mathbb{C}$. Following the proof of theorem 7.1 of [11] one can prove the next theorem.

Theorem 6.1. *If* $\lim_{n\to\infty}c_{n,n+k} = 1$ *for all* $k\in\mathbb{Z}$*, then the complex measure* $X\mapsto\lim_{|z|\to\infty}E_{|z\rangle,|z\rangle}(X)$ *is concentrated at the point* $\arg z + 2\pi l$*, where* $l\in\mathbb{Z}$ *is such that* $\arg z + 2\pi l\in[\alpha-\pi,\alpha+\pi)$.

If $\lim_{n\to\infty} c_{n,n+k} = 1$ for all $k \in \mathbb{Z}$, then

$$\lim_{|z|\to\infty} \int_{\alpha-\pi}^{\alpha+\pi} \theta^m \, \mathrm{d}E_{|z\rangle,|z\rangle}(\theta) = (\arg z + 2\pi l)^m$$

for all $m \in \mathbb{N}$ and we say that E behaves well in the classical limit (of coherent states).

The canonical phase observable E_{can} and the phase space observables $E^{|n\rangle}, n \in \mathbb{N}$, behave well in the classical limit [11]. It can be shown (see appendix G of [4]) that $\lim_{n\to\infty} c^s_{n,n+k} = 1, k \in \mathbb{N}$, if $|\lambda| \leqslant 1$ and $\lambda \neq 1$, or $\mathrm{Re}\, s \leqslant 0$.

Since $|z\rangle \in \mathcal{D}^s$ for all $z \in \mathbb{C}$ and when $s < 1$ we see that the quasi-probability density function

$$\begin{aligned} g^s_{|z\rangle}(z') &= (1-\lambda)\,\mathrm{e}^{-(1-\lambda)|z'-z|^2} \\ &= (1-\lambda)\exp(-(1-\lambda)(|z'|^2 + |z|^2 - 2|zz'|\cos(\arg z' - \arg z))) \end{aligned} \tag{33}$$

for all $z' \in \mathbb{C}$ is positive, that is, a probability density. Again, denoting $x := |z'|$ and $\theta := \arg z'$, the phase probability distribution (see also [19, 23])

$$\begin{aligned} h^s_{|z\rangle}(\theta) &:= \frac{1}{2\pi}\int_0^\infty g^s_{|z\rangle}(x\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}x^2 = \frac{1}{2\pi}\mathrm{e}^{-(1-\lambda)|z|^2} + \sqrt{1-\lambda}|z|\cos(\theta - \arg z) \\ &\quad \times \exp(-(1-\lambda)|z|^2\sin^2(\theta-\arg z))\frac{1}{2\sqrt{\pi}}\mathrm{erfc}(-\sqrt{1-\lambda}|z|\,\cos(\theta-\arg z)) \end{aligned} \tag{34}$$

which tends to a 2π-periodic Dirac δ-distribution $\delta_{2\pi}(\theta - \arg z)$ when $|z| \to \infty$, or when $s \to 1-(\lambda \to -\infty)$. Thus, if $s < 1$ all quasi phase observables E^s behave well in the classical limit of coherent states. Especially, the quasi phase observable $E^{s=0}$ of the Wigner–Weyl quantized phase angle $\Phi^{s=0}$ behaves well in the classical limit. Note that in the limit $s \to 1-$ we can formally write the angle margin of the Glauber–Sudarshan P-distribution in the form $h^{s=1}_{|z\rangle}(\theta) = \delta_{2\pi}(\theta - \arg z)$ for all $z \in \mathbb{C}$ and $z \neq 0$.

It has been noted earlier [11, 24, 25] that the *minimum variance*, or the Lévy measure, is a good measure for phase uncertainty. Let $g_n\colon \mathbb{R} \to [0,\infty]$ be a periodic probability density for all $n \in \mathbb{N}$. If $\lim_{n\to\infty} g_n(\theta) = \delta_{2\pi}(\theta - \alpha)$ then the minimum variance of g_n can be approximated by the integral

$$\int_{-\pi}^{\pi} \theta^2 g_n(\theta+\alpha)\,\mathrm{d}\theta \tag{35}$$

when $n \gg 0$ [11].

From equation (34) one sees that we may always assume that $\arg z = 0$, that is, $|z\rangle = |r\rangle$ where $r := |z|$. When $r \gg 0$ we can approximate the probability distribution $h^s_{|r\rangle}(\theta)$ by the function

$$\tilde{h}^s_{|r\rangle}(\theta) := \frac{1}{\sqrt{\pi}}\sqrt{1-\lambda}\,r\,\mathrm{e}^{(\lambda-1)r^2\theta^2} \tag{36}$$

and we see that the minimum variance of $h^s_{|r\rangle}(\theta)$ is

$$\mathrm{VAR}(E^s, |z\rangle) \sim \frac{1}{2(1-\lambda)|z|^2} = \frac{1-s}{4|z|^2}. \tag{37}$$

Using the results of [11, 26] one sees that for the canonical (or Pegg–Barnett) phase distribution

$$h^{\mathrm{can}}_{|r\rangle}(\theta) := \frac{1}{2\pi}\mathrm{e}^{-r^2}\sum_{n,m=0}^{\infty}\frac{r^{n+m}}{\sqrt{n!m!}}\mathrm{e}^{\mathrm{i}(n-m)\theta} \sim \sqrt{\frac{2}{\pi}}\,r\,\mathrm{e}^{-2r^2\theta^2} \tag{38}$$

the minimum variance is

$$\mathrm{VAR}(E^{\mathrm{can}}, |z\rangle) \sim \frac{1}{4|z|^2}. \tag{39}$$

Especially, for the Wigner phase distribution ($\lambda = -1$ or $s = 0$)

$$h^{s=0}_{|r\rangle}(\theta) \sim h^{\mathrm{can}}_{|r\rangle}(\theta) \tag{40}$$

and

$$\mathrm{VAR}(E^{s=0}, |z\rangle) \sim \mathrm{VAR}(E^{\mathrm{can}}, |z\rangle) \tag{41}$$

when $|z| \gg 0$. In references [19, 23] differences between the phase probability densities $h^{\mathrm{can}}_{|r\rangle}(\theta)$ and $h^{s=0}_{|r\rangle}(\theta)$ have been illustrated using pictures. For the phase space phase observables $E^{|k\rangle}$ one gets [11]

$$\mathrm{VAR}(E^{|k\rangle}, |z\rangle) \sim \frac{k+1}{2|z|^2} \tag{42}$$

for all $k \in \mathbb{N}$.

Since the probability density of $X \mapsto \langle r|E^{|k\rangle}(X)|r\rangle$ is of the form [11]

$$h^{|k\rangle}_{|r\rangle}(\theta) := \frac{1}{2\pi k!}\int_{r^2}^{\infty} \mathrm{e}^{-v} v^k \,\mathrm{d}v + \frac{1}{2\pi k!}\mathrm{e}^{-r^2\sin^2\theta} 2r\cos\theta \times \sum_{n=0}^{k} \binom{k}{n} (r^2\sin^2\theta)^{k-n} \int_{-r\cos\theta}^{\infty} \mathrm{e}^{-u^2} u^{2n}\,\mathrm{d}u \tag{43}$$

one gets for all $\lambda < 1$ that

$$h^{s}_{|z\rangle}(\theta) = (1-\lambda)\sum_{k=0}^{\infty} \lambda^k h^{|k\rangle}_{|z\rangle}(\theta). \tag{44}$$

Also by direct calculation

$$\mathrm{VAR}(E^{s}, |z\rangle) \sim (1-\lambda)\sum_{k=0}^{\infty} \lambda^k\, \mathrm{VAR}(E^{|k\rangle}, |z\rangle) \tag{45}$$

when $|z| \gg 0$. Thus, when $\lambda > 0$, $\lambda \approx 0$

$$h^{s}_{|z\rangle}(\theta) \approx h^{|0\rangle}_{|z\rangle}(\theta) + \lambda h^{|1\rangle}_{|z\rangle}(\theta) + \cdots$$

and

$$\mathrm{VAR}(E^{s}, |z\rangle) > \mathrm{VAR}(E^{|0\rangle}, |z\rangle)$$

when $|z| \gg 0$. Thus, in this case the POM E^s seems to be a phase observable describing an unsharp coherent state phase measurement (compared to the measured $E^{|0\rangle}$).

To conclude we note the following ordering between the minimum variances of different GOMs in large amplitude coherent states:

$$\begin{aligned}
&\mathrm{VAR}(E^{s}, |z\rangle) \to 0 \qquad s \to 1-\\
&\mathrm{VAR}(E^{s}, |z\rangle) < \mathrm{VAR}(E^{s=0}, |z\rangle) \sim \mathrm{VAR}(E^{\mathrm{can}}, |z\rangle) \qquad 0 < s < 1\\
&\mathrm{VAR}(E^{s=0}, |z\rangle) < \mathrm{VAR}(E^{s}, |z\rangle) < \mathrm{VAR}(E^{|0\rangle}, |z\rangle) \qquad -1 < s < 0\\
&\mathrm{VAR}(E^{|0\rangle}, |z\rangle) = \mathrm{VAR}(E^{s=-1}, |z\rangle) < \mathrm{VAR}(E^{s}, |z\rangle) \qquad s < -1\\
&\mathrm{VAR}(E^{s}, |z\rangle) \to \pi^2/3 \qquad s \to -\infty.
\end{aligned}$$

7. Summary

A phase shift covariant GOM

$$E(X) := \sum_{n,m=0}^{\infty} c_{n,m} \frac{1}{2\pi} \int_X e^{i(n-m)\theta} \, d\theta |n\rangle\langle m| \qquad X \in \mathcal{B}([0, 2\pi)) \tag{46}$$

is determined by a complex matrix $(c_{n,m})$ with $c_{n,n} \equiv 1$. It describes a measurement of the phase $\arg z$ of a coherent state $|z\rangle$ if the measure $E_{|z\rangle,|z\rangle}$ is a probability measure, since the natural covariance condition $E_{|e^{-i\alpha}z\rangle,|e^{-i\alpha}z\rangle}(X) = E_{|z\rangle,|z\rangle}(X \oplus \alpha)$ is now fulfilled. Also for a number state $|n\rangle$ the phase distribution

$$E_{|n\rangle,|n\rangle}(X) = \frac{1}{2\pi} \int_X d\theta \tag{47}$$

is totally random.

If the structure matrix $(c_{n,m})$ is positive semidefinite then E is a POM, and one may associate a probability measure $X \mapsto \langle\psi|E(X)\psi\rangle$ with it for all vector states $\psi \in \mathcal{H}$, $\|\psi\| = 1$. Examples of covariant POMs are the canonical phase observable E_{can} and the phase space phase observables $E^{|k\rangle}$, $k \in \mathbb{N}$.

For any covariant GOM one may define the first moment form as

$$\Phi = \pi I + \sum_{n \neq m=0}^{\infty} c_{n,m} \frac{i}{m-n} |n\rangle\langle m|. \tag{48}$$

It is not necessarily a bounded operator. If Φ is a self-adjoint operator it has a unique spectral measure F. It should be noted that Φ determines also the GOM E uniquely. Thus, if E is a covariant POM it is natural to ask which of the two POMs, E or F, is the associated phase observable of the self-adjoint operator Φ. There are some reasons which support E to be a phase observable:

- E is phase shift covariant, F is not;
- E gives a random distribution in number states, F does not;
- the support of E is always the whole phase interval $[0, 2\pi]$ whereas the support of F may be a proper subset of $[0, 2\pi]$.

If E is not positive, there are vector states ψ for which $E_{\psi,\psi}$ cannot be defined or it is not a probability measure whereas $F_{\psi,\psi}$ is always a probability measure. But F is not covariant in this case also.

The number operator N and Φ satisfy the correspondence principle. Namely, a formal relation

$$N\Phi - \Phi N = iI - i \sum_{n,m=0}^{\infty} c_{n,m} |n\rangle\langle m| \tag{49}$$

corresponds to the classical Poisson bracket

$$\{H, \phi\}_{\mathrm{PB}} = 1 - 2\pi \delta_{2\pi}(\theta). \tag{50}$$

The Cahill–Glauber s-parametrized covariant GOM E^s is determined by the structure matrix $\left(c^s_{n,m}\right)$ for all $s \in \mathbb{C}$ and $\mathrm{Re}\, s < 1$. An E^s is symmetric if and only if $s \in \mathbb{R}$ (and $s < 1$). If $0 \leqslant s < 1$ then E^s is not positive. If $s < 0$ then E^s is a normalized operator measure and it can be represented as the sum

$$E^s(X) = (1 - \lambda) \sum_{k=0}^{\infty} \lambda^k E^{|k\rangle}(X) \tag{51}$$

where $\lambda = (s+1)/(s-1)$. Such an E^s is a phase observable (POM) if $s \leqslant -1$.

The first moment form Φ^s is not a bounded operator if $0 < s < 1$. It is a bounded self-adjoint operator when $s \leqslant 0$. If $s = -1$ we say that $\Phi^{s=-1} = \Phi^{|0\rangle}$ is the antinormally ordered phase operator, and if $s = 0$ we say that $\Phi^{s=0}$ is the symmetrically ordered phase operator or the Wigner–Weyl quantized phase angle. The structure matrix $\left(c_{n,m}^{s=0}\right)$ of $E^{s=0}$ is not positive semidefinite since $c_{0,2}^{s=0} = \sqrt{2} > 1$. Thus, $\Phi^{s=0}$ is not determined by a phase observable, that is, by a phase shift covariant POM.

The Gahill–Glauber GOM E^s gives a (phase shift covariant) probability distribution

$$X \mapsto E^s_{|z\rangle,|z\rangle}(X) \tag{52}$$

in a coherent state $|z\rangle$ for all $s < 1$. Thus, E^s may represent a coherent state phase measurement although it is not a POM for all $s < 1$. Also E^s gives a random phase distribution in number states. When $|z| \to \infty$, $E^s_{|z\rangle,|z\rangle}$ tends to a probability measure concentrated at the point $\arg z$.

Acknowledgments

The author thanks Dr Pekka J Lahti and Dr Kari Ylinen for many fruitful discussions.

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